

A NEGISHI'S APPROACH TO COMPETITIVE EQUILIBRIUM WITH RISK OF DEFAULT

GAETANO BLOISE, PIETRO REICHLIN, AND MARIO TIRELLI

Current version: November 23, 2010

ABSTRACT. We study competitive equilibrium in sequential economies under limited commitment. Default induces permanent exclusion from financial markets and endogenously determined solvency constraints prevent debt repudiation. We establish Welfare Theorems under a weaker notion of constrained efficiency, inspired by Malinvaud, corresponding to the absence of welfare improving feasible redistributions over finite (though indefinite) horizons. A Negishi's Method permits to show that, for any arbitrary value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value. This method is also exploited to verify equilibrium indeterminacy.

KEYWORDS. Limited commitment; solvency constraints; Malinvaud efficiency; Welfare Theorems; Negishi's Method; indeterminacy; market collapse.

JEL CLASSIFICATION NUMBERS. D50, D52, D61, E44, G13.

CONTENTS

1. Introduction	2
2. Fundamentals	5
2.1. Time and uncertainty	5
2.2. Vector spaces	5
2.3. Individuals	5
2.4. Endowments	5
2.5. Preferences	6
2.6. Uniform impatience	6

G. Bloise, Department of Economics, University of Rome III (gaetano.bloise@uniroma3.it).

P. Reichlin, LUISS Guido Carli, Rome; CEPR; EIEF (preichlin@luiss.it).

M. Tirelli, Department of Economics, University of Rome III (tirelli@uniroma3.it).

We thank David K. Levine and Herakles M. Polemarchakis for helpful comments and observations. We are also grateful to participants to the VI Cowles Conference on General Equilibrium and Its Applications and to the XIX European Workshop on General Equilibrium Theory. We finally acknowledge financial support from the Italian Ministry of University and Research (PRIN 2008). All remaining misunderstandings, omissions and errors are our own responsibility.

2.7. Boundary conditions	6
2.8. Subjective prices	6
2.9. Feasible allocations	7
3. Equilibrium	7
4. Extended Negishi's Method	9
4.1. Malinvaud efficiency	9
4.2. Social planning	11
4.3. Multiplicity of Malinvaud optima	11
4.4. Welfare Theorems	12
5. Indeterminacy	13
6. Conclusion	15
References	15
Appendix A. Proofs	16
Appendix B. Severe Punishments	26
Appendix C. Example	27

1. INTRODUCTION

In this paper, we study debt enforcement within a large class of competitive economies under sequentially complete markets. Debtors might not deliver on their promises and debt repudiation induces permanent exclusion from market participation (Eaton and Gersovitz [16], Kehoe and Levine [18, 19], Kocherlakota [21] and Alvarez and Jermann [4, 5]). At equilibrium, default is prevented by endogenously determined debt limits, quantitative bounds specific to individuals and contingencies, which enforce the maximum expansion of risk-sharing subject to individual rationality of debt repayments (Alvarez and Jermann [4]). The risk of default limits the diversification of idiosyncratic risks at equilibrium. Comparing with Arrow-Debreu economies, incomplete risk-sharing yields predictions more in line with empirical observations: the asset pricing kernel is more volatile and more sensitive to idiosyncratic risks; safe interest rates are lower and risk premia are higher; individual consumptions are imperfectly correlated with aggregate consumption and positively correlated with individual incomes.

A great advantage of Alvarez and Jermann's [4] economy with limited commitment, compared with the more traditional paradigms of incomplete asset markets and liquidity-constrained asset markets, is analytical tractability. Indeed, autarchic reservation utilities permit an unambiguous notion of constrained efficiency, corresponding to the largest feasible risk-sharing subject to participation. Alvarez and

Jermann [4] establish a qualified First Welfare Theorem and exploit a Second Welfare Theorem to decentralize constrained efficient allocations. Thus, as in classical macroeconomic analysis, a sort of Negishi's Method delivers equilibrium restrictions on asset prices and consumptions. Nevertheless, Alvarez and Jermann [4] also show that autarchy is always an equilibrium, albeit not constrained efficient, and that equilibria might not achieve a constrained efficient diversification of risk. Constrained inefficiency occurs whenever the value of aggregate endowment is not finite, or equilibrium does not exhibit high implied interest rates, interpreted as a failure of social transversality.

We expand Alvarez and Jermann's [4] approach to possibly constrained inefficient equilibria. Beyond generality, and a merely theoretical interest, this extension is of relevance for two reasons. First, constrained inefficient equilibria exhibit lower volumes of trade and smaller risk-diversification, compared with constrained efficient equilibria. This might improve predictions of macroeconomic dynamics, given the concerns raised, among others, by Krueger and Perri [23], Cordoba [14], Krueger, Lustig and Perri [24] and Ábrahám and Cárceles-Poveda [1]. Second, purely extrinsic, or sunspot, uncertainty is ineffective at constrained efficiency, but it might amplify volatility when constrained efficiency is violated. Prices might react to spontaneous revisions of expectations, thereby propagating real disturbances, sustaining turbulences and inducing financial distresses. Simple examples by Azariadis [7] and Antinolfi, Azariadis and Bullard [6] suggest a large equilibrium indeterminacy in economies with limited commitment. A deeper understanding of the complete structure of equilibria, and of related indeterminacy, might be a progress along this line of research.

A complete characterization of equilibria is attained by developing an extended Negishi's Method (Negishi [29]). In particular, we restore Welfare Theorems for a weak form of constrained efficiency. Malinvaud, or short-run, constrained optimality corresponds to the absence of a welfare-improving feasible redistribution of risk, subject to participation, over any *finite* horizon (as in Malinvaud [27, 28], Balasko and Shell [8] and Aliprantis, Brown and Burkinshaw [3]). We show that any equilibrium is a Malinvaud constrained optimum (First Welfare Theorem) and, conversely, any Malinvaud constrained optimum can be sustained as an equilibrium for some balanced distribution of initial claims (Second Welfare Theorem). Importantly, the Second Welfare Theorem is established for equilibria without mandatory savings (*i.e.*, only debt is restricted and traders are never obliged to accumulate assets).

The extended Negishi's Method resembles the canonical approach in macroeconomics (at least since Bewley [12] and Kehoe, Levine and Romer [20]). Indeed,

Malinvaud constrained efficient allocations can be obtained by the maximization of a utilitarian social welfare function over a restricted domain. This feasible set contains all allocations, satisfying material and participation constraints, attainable by means of finitely many variations of a reference allocation. The restricted social planner program behaves as the unrestricted social planning program, providing a sort of *local* contract curve, conditional on a given reference allocation. By varying the reference allocation, we show that, *globally*, there is a vast multiplicity of constrained efficient allocations and, hence, by Welfare Theorems, of competitive equilibria. In particular, given any arbitrary value of social welfare in between autarchy and constrained optimality, there exists an equilibrium attaining that value. In other terms, there is a continuum of equilibria with welfare declining from constrained efficiency to autarchy, provided that the latter is constrained inefficient. Finally, we prove that this vast multiplicity of Malinvaud constrained optima induces indeterminacy: an economy might admit an infinite set of equilibria for a *given* distribution of initial claims (that is, given contractual obligations inherited from the unrepresented past). Yet, not all distributions of initial claims exhibit indeterminacy.

To some extent, the multiplicity of equilibria under limited commitment is not surprising, though its pervasiveness might be. Debt contracts are enforced by the threat of exclusion from financial markets and might sustain limited risk-sharing at equilibrium. However, the underlying mechanism is merely reputational and, in a sense, fragile. The value of reputation might dissipate over time and contingencies because of *dynamic complementarity* (according to the terminology of Azariadis, Antinolfi and Bullard [6]): the anticipation of tighter debt limits in the future reduces the current value of market participation and increases the incentive to default; debt limits immediately become tighter, responding to lower participation incentives, and interest rate falls to balance the reduced volume of liabilities. This self-fulfilling mechanism requires a failure of social transversality (a violation of high implied interest rates). Consequently, any institutional arrangement enforcing social transversality eliminates the real multiplicity of equilibria. This happens, for example, when a sufficiently productive asset of infinite-maturity is traded, or when outstanding public debt is backed by a non-vanishing stream of real primary public surpluses.

A deeper understanding of equilibrium properties deserves additional work. In the simple example of Azariadis, Antinolfi and Bullard [6] (see also Azariadis [7] and our appendix C), inspired by Bewley [11], there are two individuals with alternating endowments and a constant aggregate endowment. When endowments

are sufficiently dispersed, the autarchy is constrained inefficient. The economy admits a constrained efficient steady state and a continuum of other constrained inefficient equilibria converging to autarchy. Hence, any displacement from constrained efficiency induces a complete collapse of financial markets in the long-run. Examples under less restrictive hypotheses are extremely hard to handle even for constrained efficient equilibria (for instance, Ljungqvist and Sargent [26, Chapter 20.11]). However, it seems that, in general, the value of reputation might dissipate at some contingencies and preserve at other contingencies, with a complete collapse of markets occurring only with positive (and possibly extremely low) probability, in the spirit of the recent rare event doctrine for the understanding of asset prices (for instance, Barro [9]).

Is multiplicity of competitive equilibria an artifact of the particular punishment for default? Our techniques exploit a relevant feature of full exclusion, namely, the fact that reservation utilities (the private values of default) are exogenous. This, indeed, permits a dual characterization of equilibria by a Negishi's Method. The approach would straightforwardly extend to other exogenously specified reservation utilities. Multiplicity would persist under more lenient punishments, whereas it would disappear for more severe punishments. For instance, when default is punished by a partial confiscation of the endowment, beyond full exclusion, private debts are backed by a share of private endowments and Malinvaud constrained efficiency coincides with (unrestricted) constrained efficiency, thus ensuring determinacy of competitive equilibrium (see appendix B).

Partial exclusion requires an alternative approach of analysis, as the extended Negishi's Method fails because of the pecuniary externality (that is, redistributions of risk modify reservation utilities through changes in prices). Bulow and Rogoff [13] and Hellwig and Lorenzoni [17] provide a relevant instance of partial exclusion: debt repudiation inhibits future borrowing, though lending remains unrestricted. Other examples include temporary exclusion for a limited number of periods or permanent exclusion with some probability. A unified treatment of all such instances, at a level of generality comparable with that in this paper, does not seem straightforward, as an analogous dual approach is not practicable. However, the dynamic complementarity of debt constraints remains and might induce some form of failure of social transversality. Besides, other causes of indeterminacy might emerge (due to the additional pecuniary externality). For instance, Hellwig and Lorenzoni [17, Supplementary Material] provide a simple example showing local indeterminacy of a steady state equilibrium (although indeterminacy only affects a transitional phase).

The paper is organized as follows. In sections 2 and 3, we lay out the fundamentals of a general multi-agent economy with uncertainty and we define a notion of competitive equilibrium with sequential trades and not-too-tight debt constraints. Some technical aspects in section 2 might be skipped at a first reading without affecting the understanding of remaining parts of the paper. In section 4, we present the Negishi's Method, establishing Welfare Theorems under Malinvaud constrained efficiency. In section 5, we prove equilibrium indeterminacy for an null distribution of initial claims. All proofs are collected in the appendix.

2. FUNDAMENTALS

2.1. Time and uncertainty. Time and uncertainty are represented by an event-tree \mathcal{S} , a countably infinite set, endowed with partial ordering \succeq . For a date-event σ in \mathcal{S} , $t(\sigma)$ in $\mathcal{T} = \{0, 1, 2, \dots, t, \dots\}$ denotes its date and

$$\sigma_+ = \{\tau \in \mathcal{S}(\sigma) : t(\tau) = t(\sigma) + 1\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma) = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}$$

is the set of all date-events τ in \mathcal{S} (weakly) following date-event σ in \mathcal{S} . The initial date-event is ϕ in \mathcal{S} , with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every σ in \mathcal{S} ; the initial date-event in $\mathcal{S}(\sigma)$ is σ in \mathcal{S} . This construction is canonical (Debreu [15, Chapter 7]).

2.2. Vector spaces. We essentially adhere to Aliprantis and Border [2, Chapters 5-8] for terminology and notation. The reference vector space is $L = \mathbb{R}^{\mathcal{S}}$, the space of all real-valued maps on \mathcal{S} , with typical element $v = (v_\sigma)_{\sigma \in \mathcal{S}}$. The vector space L is endowed with the canonical order: an element v of L is *positive* if $v_\sigma \geq 0$ for every σ in \mathcal{S} ; it is *strictly positive* if $v_\sigma > 0$ for every σ in \mathcal{S} ; finally, it is *uniformly strictly positive* if, for some $\epsilon > 0$, $v_\sigma \geq \epsilon$ for every σ in \mathcal{S} . For a positive element v of L , we simply write $v \geq 0$ and, when v in L is also non-null, $v > 0$. An element v of L is *bounded* if, for some $\epsilon > 0$, $|v_\sigma| \leq \epsilon$ for every σ in \mathcal{S} ; it is *summable* if $\sum_{\sigma \in \mathcal{S}} |v_\sigma|$ is finite; it is *eventually vanishing* if $\{\sigma \in \mathcal{S} : |v_\sigma| > 0\}$ is a finite subset of \mathcal{S} . The vector subspace of L , consisting of all eventually vanishing elements v of L , is denoted by C . Finally, the vector space L is endowed with the product topology.

2.3. Individuals. There is a finite set \mathcal{J} of individuals. For every individual i in \mathcal{J} , the *consumption space* X^i is the positive cone of the *commodity space* L . A consumption plan x^i in X^i is *interior* (respectively, *bounded*) if it is uniformly strictly

positive (respectively, bounded). An *allocation* is a distribution of consumption plans across individuals. The space of allocations is

$$X = \{x \in L^{\mathcal{J}} : x^i \in X^i \text{ for every } i \in \mathcal{J}\}.$$

An allocation x in X is *interior* (respectively, *bounded*) if every consumption plan x^i in X^i is interior (respectively, bounded).

2.4. Endowments. For every individual i in \mathcal{J} , the *endowment* e^i in X^i is interior and bounded. In particular, there exists a sufficiently small $1 > \epsilon > 0$ satisfying, at every date-event σ in \mathcal{S} ,

$$\epsilon \leq \bigwedge_{i \in \mathcal{J}} e_{\sigma}^i \leq \bigvee_{i \in \mathcal{J}} e_{\sigma}^i \leq \frac{1}{\epsilon} \frac{1}{\text{card}(\mathcal{J})},$$

where $\text{card}(\mathcal{J})$ in \mathbb{N} denotes the cardinality of \mathcal{J} . This hypothesis imposes a uniform lower bound on the endowment of individuals and, across individuals, an upper bound on the aggregate endowment.

2.5. Preferences. We allow for heterogeneous impatience and subjective beliefs, retaining time-additivity of intertemporal utilities in order to simplify the presentation. For every individual i in \mathcal{J} , the per-period utility function $u^i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is bounded, continuous, continuously differentiable, strictly increasing and strictly concave. (As far as smoothness is concerned, more precisely, the per-period utility function is continuously differentiable on \mathbb{R}_{++} .) For every individual i in \mathcal{J} , the *utility function* $U^i : X^i \rightarrow \mathbb{R}$ is given by

$$U^i(x^i) = \sum_{\sigma \in \mathcal{S}} \pi_{\sigma}^i u^i(x_{\sigma}^i),$$

where π^i is a strictly positive summable element of L . Also, for any date-event σ in \mathcal{S} , at any consumption plan x^i in X^i ,

$$U_{\sigma}^i(x^i) = \frac{1}{\pi_{\sigma}^i} \sum_{\tau \in \mathcal{S}(\sigma)} \pi_{\tau}^i u^i(x_{\tau}^i).$$

This is the continuation utility beginning from date-event σ in \mathcal{S} .

2.6. Uniform impatience. We impose a uniform bound on the marginal rate of substitution of perpetual future consumption for current consumption. This hypothesis implies a uniform form of impatience across individuals and date-events (see, for instance, Levine and Zame [25, Assumption 5] or Santos and Woodford [30, Assumption 2]). Basically, there exists a sufficiently small $1 > \eta > 0$ satisfying, for every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} ,

$$\pi_{\sigma}^i \geq \eta \sum_{\tau \in \mathcal{S}(\sigma)} \pi_{\tau}^i.$$

2.7. Boundary conditions. This additional hypothesis ensures interiority. For every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} ,

$$\eta u^i(0) + (1 - \eta) u^i\left(\frac{1}{\epsilon}\right) < u^i(\epsilon),$$

where $1 > \epsilon > 0$ is given by the bounds on endowments and $1 > \eta > 0$ by the hypothesis of uniform impatience.

2.8. Subjective prices. At an interior consumption plan x^i in X^i , the subjective price p^i in P^i is defined by

$$(p^i_{\sigma})_{\sigma \in \mathcal{S}} = (\pi^i_{\sigma} \partial u^i(x^i_{\sigma}))_{\sigma \in \mathcal{S}}.$$

The subjective price p^i in P^i is a strictly positive summable element of L .

2.9. Feasible allocations. An allocation x in X is *feasible* if it exhausts aggregate resources and satisfies participation constraints, that is,

$$\sum_{i \in \mathcal{J}} x^i = \sum_{i \in \mathcal{J}} e^i$$

and, for every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} ,

$$U^i_{\sigma}(x^i) \geq U^i_{\sigma}(e^i).$$

The space of all feasible allocations is denoted by $X(e)$. Notice that feasibility reflects both material constraints and participation constraints.

Under the maintained assumptions on preferences and endowments, every feasible allocation is, as a matter of fact, an interior allocation. The particular form of boundary conditions, which is a joint restriction on preferences and endowments, guarantees interiority of consumptions, subject to participation constraints, avoiding unbounded per-period utilities and, hence, simplifying the presentation.

Lemma 1 (Interiority). *Every feasible allocation is interior.*

3. EQUILIBRIUM

Trade occurs sequentially. In every period, a full spectrum of elementary Arrow securities is available for trade, each of which promising a unitary payoff, contingent on the occurrence of a distinct event in the following period. The asset market is, thus, sequentially complete. It simplifies to represent implicit prices of contingent commodities in terms of present values. They are denoted by p in P , the space of all strictly positive elements of L . At every date-event σ in \mathcal{S} , a portfolio, with

deliveries v in L at the following date-events, has a market value, in terms of current consumption, given by

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau.$$

It should be remarked that, at a price p in P , the present value of an arbitrary bounded consumption plan is not necessarily finite.

An individual i in \mathcal{I} participates into financial markets. The holding of securities is represented by a *financial plan* v^i in V^i , the space of all unrestricted elements of L . Positive values correspond to claims, whereas negative values are liabilities. This participation occurs subject to a *sequential budget constraint* imposing, at every date-event σ in \mathcal{S} ,

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau^i + p_\sigma (x_\sigma^i - e_\sigma^i) \leq p_\sigma v_\sigma^i.$$

Accumulated wealth serves to finance current consumption, in excess to current endowment, and current net asset positions (claims or liabilities). Participation into financial markets is further restricted by quantitative limits to private liabilities. These *debt limits* are given by f^i in F^i , the set of all *positive* and *bounded* elements of L . The financial plan v^i in V^i is subject, at every date-event σ in \mathcal{S} , to a *debt (or solvency) constraint* of the form

$$-f_\sigma^i \leq v_\sigma^i.$$

From the perspective of the individual, debt limits are given exogenously.

As in Eaton and Gersovitz [16], Kehoe and Levine [18], Kocherlakota [21] and Alvarez and Jermann [4], commitment is limited. Individuals might not honor their debt obligations, even though the material availability of future endowments would suffice for a complete repayment. When debt is repudiated, assets are seized and the individual is excluded from future participation into financial markets, though maintaining claims into future uncertain endowment. Thus, unhonored debt induces a permanent reverse to autarchy. At equilibrium, debt limits serve to guarantee that, on the one side, debt repudiation is not profitable for individuals and, on the other side, the maximum sustainable development of financial markets is enforced. This is the notion of equilibrium with *not-too-tight* debt constraints provided by Alvarez and Jermann [4].

Formally, an allocation x in X is an *equilibrium allocation* if there exist a price p in P , debt limits f in F and financial plans v in V satisfying the following properties:

- (a) For every individual i in \mathcal{I} , the plan (x^i, v^i) in $X^i \times V^i$ is optimal subject to budget and debt constraints, given initial claims, that is, it maximizes

intertemporal utility subject, at every date-event σ in \mathcal{S} , to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma (\bar{x}_\sigma^i - e_\sigma^i) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$-(\bar{v}_\tau^i + f_\tau^i)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth v_ϕ^i in \mathbb{R} .

(b) Commodity and financial markets clear, that is,

$$\sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} e^i \text{ and } \sum_{i \in \mathcal{I}} v^i = 0.$$

(c) For every individual i in \mathcal{I} , debt limits are not-too-tight, that is, at every date-event $\bar{\sigma}$ in \mathcal{S} ,

$$J_{\bar{\sigma}}^i(-f_{\bar{\sigma}}^i; f^i) = U_{\bar{\sigma}}^i(e^i),$$

where

$$J_{\bar{\sigma}}^i(w_{\bar{\sigma}}^i; f^i) = \sup U_{\bar{\sigma}}^i(\bar{x}^i)$$

subject, at every date-event σ in $\mathcal{S}(\bar{\sigma})$, to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma (\bar{x}_\sigma^i - e_\sigma^i) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$-(\bar{v}_\tau^i + f_\tau^i)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth $w_{\bar{\sigma}}^i$ in \mathbb{R} . (By convention, the supremum over an empty set is negative infinity.)

Notice that, at equilibrium, for every individual i in \mathcal{I} , at every date-event σ in \mathcal{S} ,

$$U_\sigma^i(x^i) = J_\sigma^i(v_\sigma^i; f^i) \geq J_\sigma^i(-f_\sigma^i; f^i) = U_\sigma^i(e^i).$$

Hence, an equilibrium allocation x in X is, as a matter of fact, an element of $X(e)$, the space of feasible allocations.

We adopt a restrictive notion of equilibrium: first, we require debt limits to be positive; second, we exclude speculative bubbles. Possibly negative debt limits, which are allowed by Alvarez and Jermann [4], would constrain individuals to hold positive wealth at some contingencies; by defaulting at those contingencies, a trader would in fact refuse a net payment; the enforcement mechanism would be unnatural: A trader would certainly profit from a *voluntary* exclusion from markets after receiving (and, hence, accepting) the net payment. Furthermore, negative, and possibly unbounded, debt limits would sustain speculative bubbles at equilibrium,

a circumstance that has been shown by Kocherlakota [22], based on some properties of homogeneity of the budget set. Finally, notice that debt limits are *consistent* at equilibrium (according to the terminology borne out by Levine and Zame [25]). Debt limits are consistent when the maximum amount of debt can be honored by means of current endowment and by issuing future contingent debt up to the limit, that is, for every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} ,

$$p_\sigma f_\sigma^i \leq p_\sigma e_\sigma^i + \sum_{\tau \in \sigma_+} p_\tau f_\tau^i.$$

4. EXTENDED NEGISHI'S METHOD

4.1. Malinvaud efficiency. Malinvaud efficiency is inherited from studies on capital theory (*e.g.*, Malinvaud [27, 28]) and overlapping generations economies (*e.g.*, Balasko and Shell [8]). The canonical notion of Pareto efficiency requires the absence of a welfare improvement, subject to material and participation constraints. Thus, an allocation x in $X(e)$ is *Pareto (constrained) efficient* if it is not Pareto dominated by an alternative allocation z in $X(e)$. The notion of Malinvaud efficiency, instead, imposes weaker restrictions, as it simply requires the absence of a welfare improvement, subject to material and participation constraints, over any arbitrary finite horizon. Consistently, an allocation x in $X(e)$ is *Malinvaud (constrained) efficient* if it is not Pareto dominated by an alternative allocation z in $C(e, x)$, where

$$C(e, x) = \left\{ z \in X(e) : \sum_{i \in \mathcal{J}} |z^i - x^i| \in C \right\}$$

is the set of all allocations z in $X(e)$ that modify allocation x in $X(e)$ only over a finite horizon. (Remember that C is the set of all eventually vanishing elements of L .) Clearly, any Pareto optimum is a Malinvaud optimum. However, Malinvaud optimality is a largely weaker requirement: for instance, any autarchic allocation is a Malinvaud optimum. Indeed, as only variations of consumption over finite horizons are allowed, no redistribution satisfies participation constraints at terminal nodes of the finite horizon, because a donor cannot be compensated by the promise of higher continuation utilities. Hence, by induction, no redistribution is the only feasible allocation.

Malinvaud efficiency admits a characterization in terms of supporting price. This is an elaboration on the common duality argument, developed in the literature on capital theory and, more recently, for economies of overlapping generations by Aliprantis, Brown and Burkinshaw [3]. The (algebraic) dual of the vector subspace C of L can be identified with L itself, under the duality operation given, for every

(v, f) in $C \times L$, by

$$f(v) = f \cdot v = \sum_{\sigma \in \mathcal{S}} f_{\sigma} v_{\sigma}.$$

To simplify notation, let

$$C^*(e, x) = \left\{ z \in X^*(e) : \sum_{i \in \mathcal{I}} |z^i - x^i| \in C \right\},$$

where $X^*(e)$ is the set of all allocations z in X such that, for every individual i in \mathcal{I} , at every date-event σ in \mathcal{S} ,

$$U_{\sigma}^i(z^i) \geq U_{\sigma}^i(e^i).$$

Lemma 2 (First-order conditions). *An allocation x in $X(e)$ is Malinvaud efficient if and only if there exists a price p in P satisfying, at every allocation z in $C^*(e, x)$, for every individual i in \mathcal{I} ,*

$$(s) \quad U^i(z^i) > U^i(x^i) \text{ only if } p \cdot (z^i - x^i) > 0.$$

Equivalently, an allocation x in $X(e)$ is Malinvaud efficient if and only if there exists a price p in P satisfying, for every individual i in \mathcal{I} , at every date-event σ in \mathcal{S} ,

$$(C-1) \quad \left(\frac{p_{\tau}}{p_{\sigma}} \right)_{\tau \in \sigma_+} \geq \left(\frac{p_{\tau}^i}{p_{\sigma}^i} \right)_{\tau \in \sigma_+}$$

and

$$(C-2) \quad \sum_{\tau \in \sigma_+} \left(\frac{p_{\tau}}{p_{\sigma}} \right) (U_{\tau}^i(x^i) - U_{\tau}^i(e^i)) = \sum_{\tau \in \sigma_+} \left(\frac{p_{\tau}^i}{p_{\sigma}^i} \right) (U_{\tau}^i(x^i) - U_{\tau}^i(e^i)),$$

where p^i in P^i is the subjective price at interior consumption plan x^i in X^i .

Restriction (s) coincides with an admittedly abstract characterization of Malinvaud optima in terms of supporting positive linear functionals, whereas conditions (C-1)-(C-2) uncover an equivalent formulation in terms of more treatable first-order conditions. For the sake of simplicity, though this is unprecise, the above characterization might be illustrated by referring to a canonical social planner problem. Restrictions (C-1)-(C-2) correspond, in this analogy, to the Euler equations induced by the maximization of (weighted) social welfare subject to material constraints and to participation constraints. They basically rule out the circumstance of a constrained individual exhibiting a marginal rate of substitution strictly above the marginal rate of substitution of an unconstrained individual. This, indeed, would expose to an arbitrage opportunity, as a substitution of future consumption for current consumption of the unconstrained individual, balanced by the opposite

substitution for the constrained individual, would not violate participation constraint, as utility of the unconstrained individual is strictly above the autarchic utility, and would produce a welfare improvement.

The remarkable implication of this full characterization is that a Malinvaud optimum does not impose any restriction in terms of social transversality or, alternatively, does not rule out any arbitrage opportunity *at infinitum*. A substitution of current consumption for perpetual future consumption might still generate a welfare improvement, subject to feasibility.

4.2. Social planning. A canonical method in macroeconomic theory permits the study of Pareto efficient allocations by means of a social planning programme. Given welfare weights θ in Θ , social welfare is measured by the weighted sum of utilities,

$$W_\theta(x) = \sum_{i \in \mathcal{I}} \theta^i U^i(x^i),$$

where

$$\Theta = \left\{ \theta \in \mathbb{R}_+^{\mathcal{I}} : \sum_{i \in \mathcal{I}} \theta^i = 1 \right\}.$$

It is well-established that an allocation x in $X(e)$ is Pareto efficient if and only if there exist welfare weights θ in Θ satisfying

$$W_\theta(x) = \max_{z \in X(e)} W_\theta(z).$$

Various properties of efficient allocations are directly obtained by exploiting this well-behaved convex programme.

To provide an analogous treatment of Malinvaud efficiency, we introduce a restricted feasible set for social planning. Given an allocation x in $X(e)$, define

$$F(e, x) = \text{closure} \left\{ z \in X(e) : \sum_{i \in \mathcal{I}} |z^i - x^i| \in C \right\},$$

where the closure is taken in the product topology. Intuitively, this restricted feasible set consists of all allocations z in $X(e)$ that can be approached by means of finitely many variations of a reference allocation x in $X(e)$. Importantly, this restricted feasible set is convex and compact.

Lemma 3 (Social planning). *An allocation x in $X(e)$ is Malinvaud efficient if and only if, for some allocation y in $X(e)$ such that x lies in $F(e, y)$, there exist welfare weights θ in Θ satisfying*

$$W_\theta(x) = \max_{z \in F(e, y)} W_\theta(z).$$

The comparison between the Pareto (unrestricted) planner problem and the Malinvaud (restricted) planner problem reveals strong analogies. By a canonical application of the Maximum Theorem, solutions vary continuously with welfare weights. Hence, the restricted planner problem yields a connected set of Malinvaud efficient allocations for any arbitrarily given reference feasible allocation. This, along with Welfare Theorems, provides a potential tool for the computation of competitive equilibria.

4.3. Multiplicity of Malinvaud optima. We now provide a partial characterization of Malinvaud optima. In particular, we prove that there exists a continuum of such optima with social welfare decreasing from Pareto efficiency to autarchy. (Obviously, when the autarchy is Pareto efficient, this multiplicity disappears.) Malinvaud optima are parameterized by welfare weights θ in Θ and an index ξ in $\Xi = [0, 1]$ measuring the failure of Pareto (constrained) optimality. Hence, the set of Malinvaud optima contains a set that is isomorphic to $\Theta \times \Xi$.

Proposition 1 (Multiplicity). *Given welfare weights θ in Θ , for any arbitrary value ξ in $\Xi = [0, 1]$, there exists a Malinvaud efficient allocation x in $X(e)$ with social welfare satisfying*

$$(w-1) \quad W_{\theta}(x) = W_{\theta}(e) + \xi \left(\max_{z \in X(e)} W_{\theta}(z) - W_{\theta}(e) \right)$$

and

$$(w-2) \quad W_{\theta}(x) = \max_{z \in F(e, x)} W_{\theta}(z).$$

This simple characterization emerges by means of artificial truncated planner problems, along with a limit argument. A truncation consists in imposing additional restrictions on the amount of redistributed resources that can be implemented out of some finite horizon. For a given truncation, the severity of these additional restrictions determines the value of the social planner problem: under the most severe restrictions, the redistribution vanishes out of a finite horizon and, hence, the autarchy is the only feasible allocation (indeed, a decrease of consumption in the last period of the truncation cannot be compensated by an increase of consumption in the following periods and, hence, by induction, no redistribution is the only feasible policy); under the least severe restrictions, any feasible allocation can be implemented and, hence, a Pareto optimum obtains. It follows that, for any given truncation, some properly chosen degree of severity of additional constraints would yield a given social welfare in between autarchy and Pareto efficiency. Taking the limit over finite horizons, a limit allocation emerges with a given social welfare value (as this can be assumed to be constant along the sequence). This limit allocation

is Malinvaud efficient because, as the finite horizon extends along the sequence of truncations, first-order conditions are satisfied along larger and larger horizons.

4.4. Welfare Theorems. We here show equivalence between equilibrium allocations and Malinvaud efficient allocations. Indeed, any equilibrium allocation is Malinvaud efficient (First Welfare Theorem) and any Malinvaud efficient allocation emerges as an equilibrium allocation for some balanced distribution of initial claims (Second Welfare Theorem). As a matter of fact, we prove that Malinvaud efficiency exhausts all restrictions on equilibrium prices and allocations.

Proposition 2 (First Welfare Theorem). *Any equilibrium allocation is a Malinvaud efficient allocation.*

The First Welfare Theorem is almost immediate. Indeed, first-order conditions for a Malinvaud optimum coincides with those for an equilibrium under limited commitment (see Alvarez and Jermann [4]). At equilibrium, the marginal rate of substitution of an individual falls below the market rate of substitution only if this individual is constrained in issuing further debt obligations, for otherwise a budget-balanced (marginal) substitution of future consumption for current consumption would yield an increase in welfare.

Proposition 3 (Second Welfare Theorem). *Any Malinvaud efficient allocation is an equilibrium allocation.*

The proof of the Second Welfare Theorem cannot rely on a traditional separation argument alone. Indeed, separation yields potential equilibrium prices fulfilling first-order conditions (lemma 2). Such prices, however, might not belong to the dual of the commodity space (restricted by the aggregate endowment) and, thus, might not deliver a well-defined intertemporal accounting. In order to provide their Second Welfare Theorem for Pareto efficient allocations, Alvarez and Jermann [4] assume that prices belong to the dual of the (restricted) commodity space (the hypothesis of high implied interest rates) and recover financial plans at equilibrium as the present value of future contingent net trades. We cannot count on this simple method and need an alternative proof. Furthermore, differently from Alvarez and Jermann [4], as well as from Kocherlakota [22], we impose positivity of debt limits (individuals cannot be restricted to hold positive amounts of wealth along the infinite horizon), which poses additional difficulties.

To recover financial plans, we construct an adjustment process that increases debt, when more debt is budget-feasible, and decreases debt, when outstanding debt is budget-unfeasible. This process admits a fixed point and, at the fixed point,

sequential budget constraints are balanced and financial markets clear. To identify a suitable interval, we move from a basic observation. We evaluate welfare gains, with respect to the autarchic utility, in terms of current consumption. Participation guarantees that these welfare gains are positive across date-events. Also, they fulfill sequential budget constraints at subjective prices (marginal utilities). As market rates of substitution differ from individual marginal rates of substitution only when welfare gains vanish, the process of welfare gains also satisfies sequential budget constraints at market prices. This yields an upper bound on the amount of wealth held at equilibrium, as welfare gains are positive (hence, fulfil debt limits) and sustains the given consumption plan subject to sequential budget constraints. Since financial plans need be balanced at equilibrium across individuals, the negative of the sum of welfare gains poses a lower bound to financial plans.

Optimality of consumption plans, subject to budget constraints and debt constraints, is ensured by first-order conditions at a Malinvaud optimum. Hence, it only remains to reconstruct suitable debt limits. Here, we follow Alvarez and Jermann [4]. When an individual is at the autarchic utility, outstanding debt coincides with the maximum amount of debt. When an individual is not at the autarchic utility, we compute the maximum amount of sustainable debt, which depends on the future contingent plan for debt limits. Beginning with sufficiently large debt limits, this process of adjustment generates a decreasing sequence of debt limits and, in the limit, we obtain not-too-tight debt constraints. The identification of suitable upper bounds requires some elaboration.

5. INDETERMINACY

Indeterminacy pertains to the existence of an infinite set of competitive equilibria for given endowments and *given (balanced) distribution of initial claims*. It is preliminarily worth remarking that initial claims, or liabilities, are real and that debt limits are restricted to be positive at equilibrium. Therefore, indeterminacy is not a nominal phenomenon, caused by the unavoidable multiplicity of the price level, as in monetary overlapping generations economies. Furthermore, it is not due to the occurrence of bubbles, uncovered by Kocherlakota [22], which would require that traders be forced to accumulate assets by negative debt limits (*i.e.*, mandatory savings). As a matter of fact, unrestricted debt limits would immediately induce indeterminacy by homogeneity of budget sets, independently of the established multiplicity of Malinvaud optima: a given equilibrium allocation would be consistent with any arbitrary distribution of initial claims.

The intuition for indeterminacy of competitive equilibrium is given by the fundamental characterization of Malinvaud optima (proposition 1). Indeed, welfare weights account for a merely distributive multiplicity, reflecting the distribution of initial claims. It remains an autonomous degree of multiplicity, interpretable as a measure of the reputation of debtors, varying from values enforcing the development of financial markets (efficiency) to the complete collapse of financial markets (autarchy). Beyond intuition, common techniques of analysis in the literature are not directly applicable and a formal proof is not straightforward.

Consider the space of balanced distributions of initial wealth,

$$\Omega = \left\{ \omega \in \mathbb{R}^{\mathcal{J}} : \sum_{i \in \mathcal{J}} \omega^i = 0 \right\}.$$

Also, let $E_{\omega}(e)$ denote the set of equilibrium allocations x in $X(e)$ which are consistent with the distribution of initial claims ω in Ω . By means of the adapted Negishi's Method, we establish the existence of equilibria with a null initial distribution (neither claims nor liabilities inherited from previous contractual arrangements).

Lemma 4 (Existence). *Given any arbitrary allocation y in $X(e)$,*

$$\Omega(e, y) = \{\omega \in \Omega : E_{\omega}(e) \cap M(e, y) \text{ is non-empty}\}$$

is a connected set containing the null distribution of initial claims in Ω , where

$$M(e, y) = \left\{ x \in F(e, y) : W_{\theta}(x) = \max_{z \in F(e, y)} W_{\theta}(z) \text{ for some } \theta \in \Theta \right\}.$$

The argument unfolds as follows. Within the given component of Malinvaud efficient allocations, every allocation is consistent with some distribution of initial claims by the established Second Welfare Theorem. We adjust welfare weights inversely with the initial claims held by traders. This process admits a fixed point and, at the fixed point, the initial claim of a trader is strictly positive only if the corresponding welfare weight vanishes and, thus, the trader is at the reservation utility. However, this cannot occur, as market participation, subject to positive debt limits, always guarantees a utility strictly higher than autarchy when a trader begins with a strictly positive claim.

This persistence of the null distribution, across components of Malinvaud efficient allocations, reveals the existence of an infinite set of competitive equilibria, with social welfare varying from efficiency to autarchy.

Proposition 4 (Indeterminacy). *Given the null distribution of initial claims, there exists an infinite set of equilibria, i.e., $E_0(e)$ is not finite.*

To ascertain determinacy, or indeterminacy, of equilibrium for given non-null distributions of initial claims appears technically difficult (in particular, when the supporting welfare weights are extreme points and some individuals are at the autarchic utilities). By means of numerical computations, in the classical example with periodic endowments (Azariadis [7] and Azariadis, Antinolfi and Bullard [6]), we show that the Pareto efficient steady state distribution of initial claims is *not* consistent with inefficient allocations converging to the autarchy: debt limits should be negative in some periods to sustain such allocations as equilibria (see appendix C). Thus, at least for some specifications of utility and endowments, the stationary Pareto efficient equilibrium is determinate.

6. CONCLUSION

We have shown multiplicity, and related indeterminacy, of equilibria in economies with limited enforcement and not-too-tight debt limits. In particular, we have developed a method that exploits Welfare Theorems for deriving a full characterization of equilibria. These theorems are established for a weak form of optimality, corresponding to the absence of a feasible Pareto improving redistribution over a finite number of time periods. These weak optima, in turn, are characterized by means of sequences of planning objectives with limited amounts of redistributions in the long-run. The method shows that, at equilibrium, social welfare varies from two extreme outcomes: constrained Pareto optimality and autarchy.

Our contribution bears very important consequences on the understanding of the type of equilibria that may emerge in economies where contract enforcement is limited and the no default option is implemented by imposing individual specific debt constraints. In particular, these equilibria suffer from a severe form of fragility: a change in expectations at any given equilibrium, where asset trades guarantee an optimal amount of consumption smoothing across states and time periods, might generate a contraction of net trades, in some cases leading to financial collapse.

REFERENCES

- [1] A. Ábrahám and E. Cárceles-Poveda. Endogenous trading constraints with incomplete asset markets. *Journal of Economic Theory*, 145, 974-1004, 2010.
- [2] C.D. Aliprantis and K.C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag, 1999.
- [3] C.D. Aliprantis, D.J. Brown and O. Burkinshaw. Valuation and optimality in the overlapping generations model. *International Economic Review*, 31, 275-288, 1990.
- [4] F.A. Alvarez and U.J. Jermann. Efficiency, equilibrium, and asset pricing with risk of default. *Econometrica*, 68, 775-797, 2000.

- [5] F.A. Alvarez and U.J. Jermann. Quantitative asset pricing implications of endogenous solvency constraints. *Review of Financial Studies*, 14, 1117-1151, 2001.
- [6] G. Antinolfi, C. Azariadis and J.B. Bullard. Monetary policy as equilibrium selection. *Federal Reserve Bank of St. Louis Review*, 89, 331-341, 2007.
- [7] C. Azariadis. Financial Frictions in Dynamic Economics (Class Notes), mimeograph, 2008.
- [8] Y. Balasko and K. Shell. The overlapping generations model, I: The case of pure exchange without money. *Journal of Economic Theory*, 23, 281-306, 1980.
- [9] R. Barro. Rare disasters, asset prices, and welfare costs. *The American Economic Review*, 99, 243-264, 2009.
- [10] L. Benveniste and J. Scheinkman. On the differentiability of the value function in dynamic models of economics. *Econometrica*, 47, 727-732, 1979.
- [11] T.F. Bewley. The optimum quantity of money. In J.H. Kareken and N. Wallace (eds.), *Models of Monetary Economies*, Minneapolis: Federal Reserve Bank, 1980.
- [12] T.F. Bewley. The permanent income hypothesis and long-run economic stability. *Journal of Economic Theory*, 22, 377-394, 1980.
- [13] J. Bulow and K. Rogoff. Sovereign debt: Is to forgive to forget? *American Economic Review*, 79, 43-50, 1989.
- [14] J.-C. Cordoba. U.S. inequality: Debt constraints or incomplete asset markets? *Journal of Monetary Economics*, 55, 350-364, 2008.
- [15] G. Debreu. *Theory of Value*. New York: Wiley, 1959.
- [16] J. Eaton and M. Gersovitz. Debt with potential repudiation. *Review of Economic Studies*, 48, 289-309, 1981.
- [17] M. Hellwig and G. Lorenzoni. Bubbles and self-enforcing debt. *Econometrica*, forthcoming, 2009.
- [18] T.J. Kehoe and D.K. Levine. Debt-constrained asset markets. *Review of Economic Studies*, 60, 865-888, 1993.
- [19] T.J. Kehoe and D.K. Levine. Liquidity constrained markets versus debt constrained markets. *Econometrica*, 69, 575-598, 2001.
- [20] T.J. Kehoe, D.K. Levine and P.M. Romer. On characterizing equilibria of economies with externalities and taxes as solutions to optimization problems. *Economic Theory*, 2, 43-68, 1992.
- [21] N.R. Kocherlakota. Implications of efficient risk sharing without commitment. *Review of Economic Studies*, 63, 595-609, 1996.
- [22] N.R. Kocherlakota. Injecting rational bubbles. *Journal of Economic Theory*, 142, 218-232, 2008.
- [23] D. Krueger and F. Perri. Does income inequality lead to consumption inequality? Evidence and theory. *Review of Economic Studies*, 73, 163-193, 2006.
- [24] D. Krueger, H. Lustig and F. Perri. Evaluating asset pricing models with limited commitment using household consumption data. *Journal of the European Economic Association*, 6, 715-726, 2008.
- [25] D.K. Levine and W.R. Zame. Debt constraints and equilibrium in infinite horizon economies with incomplete markets. *Journal of Mathematical Economics*, 26, 103-131, 1996.
- [26] L. Ljungqvist and T.J. Sargent. *Recursive Macroeconomic Theory*. MIT Press, second edition, 2004.

- [27] E. Malinvaud. Capital accumulation and efficient allocation of resources. *Econometrica*, 21, 233-268, 1953.
- [28] E. Malinvaud. Efficient capital accumulation: A corrigendum. *Econometrica*, 30, 570-573, 1962.
- [29] T. Negishi. Welfare economics and existence of equilibrium for a competitive economy. *Metroeconomica*, 12, 92-97, 1960.
- [30] M. Santos and M. Woodford. Rational asset pricing bubbles. *Econometrica*, 64, 19-57, 1997.

APPENDIX A. PROOFS

Proof of lemma 1. At a feasible allocation, for every individual i in \mathcal{J} , participation constraints impose, at every date-event σ in \mathcal{S} ,

$$\begin{aligned} u^i(x_\sigma^i) - u^i\left(\frac{1}{\epsilon}\right) + \frac{1}{\pi_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} \pi_\tau^i u^i\left(\frac{1}{\epsilon}\right) &\geq \\ U_\sigma^i(x^i) &\geq U_\sigma^i(e^i) \\ &\geq \frac{1}{\pi_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} \pi_\tau^i u^i(\epsilon). \end{aligned}$$

Therefore, exploiting uniform impatience and boundary conditions,

$$u^i(x_\sigma^i) \geq u^i\left(\frac{1}{\epsilon}\right) + \frac{1}{\eta} \left(u^i(\epsilon) - u^i\left(\frac{1}{\epsilon}\right) \right) > u^i(0),$$

which produces a uniformly strictly positive lower bound on consumptions. \square

Proof of lemma 2. Sufficiency of a supporting price p in P (i.e., condition (s)) for Malinvaud efficiency is obvious, as it is proved by the traditional argument for the canonical First Welfare Theorem. Therefore, we show that restrictions (c-1)-(c-2) imply condition (s). Consider any alternative allocation z in $C^*(e, x)$ and suppose that, for some individual i in \mathcal{J} ,

$$0 < U^i(z^i) - U^i(x^i) \leq \sum_{\sigma \in \mathcal{S}} p_\sigma^i (z_\sigma^i - x_\sigma^i).$$

Define, at every date-event σ in \mathcal{S} ,

$$v_\sigma^i = \frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i (z_\tau^i - x_\tau^i).$$

Notice that v^i is an element of C . A simple decomposition yields, at every date-event σ in \mathcal{S} ,

$$(*) \quad \sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + p_\sigma^i (z_\sigma^i - x_\sigma^i) \geq p_\sigma^i v_\sigma^i.$$

Furthermore, notice that convexity of preferences and participation constraints imply that, at every date-event σ in \mathcal{S} ,

$$v_\sigma^i < 0 \text{ only if } U_\sigma^i(e^i) \leq U_\sigma^i(z^i) < U_\sigma^i(x^i).$$

Therefore, restrictions (C-1)-(C-2), along with inequality (*), guarantee that, at every date-event σ in \mathcal{S} ,

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau^i + p_\sigma (z_\sigma^i - x_\sigma^i) \geq p_\sigma v_\sigma^i.$$

Consolidating across date-events, and noticing that v^i is an element of C , one obtains

$$\begin{aligned} p \cdot (z^i - x^i) &= \sum_{\sigma \in \mathcal{S}} p_\sigma (z_\sigma^i - x_\sigma^i) \\ &\geq p_\phi v_\phi^i \\ &= \left(\frac{p_\phi}{p_\phi^i} \right) \sum_{\sigma \in \mathcal{S}} p_\sigma^i (z_\sigma^i - x_\sigma^i) \\ &> 0, \end{aligned}$$

thus proving the claim.

Assume now that the allocation x in $X(e)$ is Malinvaud-efficient and define a price p in P by means, at every date-event σ in \mathcal{S} , of

$$\left(\frac{p_\tau}{p_\sigma} \right)_{\tau \in \sigma_+} = \bigvee_{i \in \mathcal{J}} \left(\frac{p_\tau^i}{p_\sigma^i} \right)_{\tau \in \sigma_+}.$$

This price p in P obviously satisfies condition (C-1). The necessity of condition (C-2) straightforwardly obtains by means of the argument in Alvarez and Jermann [4, Proposition 3.1]. As conditions (C-1)-(C-2) imply restriction (s), this completes the proof. \square

Proof of lemma 3. Preliminarily observe that an allocation x in $X(e)$ is Malinvaud inefficient if and only if it is *strictly* Pareto dominated by an alternative allocation z in $C(e, x)$. Indeed, any balanced redistribution of initial consumptions does not affect future participation. Suppose that allocation x in $X(e)$ is Malinvaud efficient. Obviously, x lies in $F(e, x)$. A canonical application of the Separation Theorem yields the existence of welfare weights θ in Θ such that

$$W_\theta(x) = \max_{z \in F(e, x)} W_\theta(z).$$

To prove the reverse claim, it suffices to establish that $C(e, x)$ is contained in $F(e, y)$. Let z be in $C(e, x)$ and consider any sequence $\{x^n\}_{n \in \mathbb{N}}$ in $C(e, y)$ approaching x in $F(e, y)$. For every individual i in \mathcal{J} , let \mathcal{F}^i be the finite subset of all date-events in \mathcal{S} at which the reallocation is not terminated, that is,

$$\sigma \in (\mathcal{S}/\mathcal{F}^i) \text{ if and only if } (z_\tau^i)_{\tau \in \mathcal{S}(\sigma)} = (x_\tau^i)_{\tau \in \mathcal{S}(\sigma)}.$$

Given $1 > \lambda > 0$, for every sufficiently large n in \mathbb{N} , consider the allocation $x^n + \lambda(z - x)$ in X . This is a balanced allocation, as consumptions are strictly positive. We show that, for every sufficiently large n in \mathbb{N} , it also satisfies participation. Notice that participation might be violated only on the finite set

$$\mathcal{F} = \bigcup_{i \in \mathcal{I}} \mathcal{F}^i.$$

If it is violated, there exists an individual i in \mathcal{I} such that, at some date-event σ in \mathcal{F}^i , for infinitely many n in \mathbb{N} ,

$$U_{\sigma}^i(x^{in} + \lambda(z^i - x^i)) < U_{\sigma}^i(e^i).$$

In the limit, this implies

$$U_{\sigma}^i(x^i + \lambda(z^i - x^i)) \leq U_{\sigma}^i(e^i),$$

which, by convexity of preferences, reveals a contradiction. Hence, for every sufficiently large n in \mathbb{N} , allocation $x^n + \lambda(z - x)$ belongs to $C(e, y)$. By closure, for every $1 > \lambda > 0$, $x + \lambda(z - x)$ lies in $F(e, y)$ and, hence, z lies in $F(e, y)$, thus proving the claim. \square

Proof of proposition 1. The proof is decomposed in several separate steps. First, we construct a sequence of truncated planner problems, by adding additional auxiliary constraints on the transfers across individuals; truncated optima exist and, at given welfare weights θ in Θ , social welfare might be measured by ξ in Ξ by controlling for the severity of additional constraints on transfers. Second, we generate a sequence of truncated optima, maintaining a constant value of social welfare, and we consider the limit allocation of these truncated planner problems. Third, we prove that the limit allocation is in fact a Malinvaud optimum fulfilling conditions (w-1)-(w-2).

Truncation. Given any t in \mathcal{T} , consider a collection of t -truncated planner problems:

$$V_{\theta}^t(\epsilon) = \max_{x \in X(e)} \sum_{i \in \mathcal{I}} \theta^i U^i(x^i)$$

subject to, at every date-event σ in $(\mathcal{S}/\mathcal{S}^t)$,

$$(\dagger) \quad \sum_{i \in \mathcal{I}} |x_{\sigma}^i - e_{\sigma}^i| \leq \epsilon,$$

where, for every t in \mathcal{T} ,

$$\mathcal{S}^t = \{\sigma \in \mathcal{S} : t(\sigma) \leq t\}.$$

Constraints are given as a continuous correspondence of ϵ in \mathbb{R}_+ with non-empty convex and compact values. (Indeed, notice that the map $x \mapsto |x|$ is convex. In addition, if allocation x in $X(e)$ satisfies constraints (\dagger) at ϵ in \mathbb{R}_{++} , then allocation

$$x - \left(\frac{\epsilon - \epsilon^*}{\epsilon} \right)^+ (x - e) \in X(e)$$

satisfies constraints (\dagger) at ϵ^* in \mathbb{R}_+ .) Hence, by the Maximum Theorem, the maximum is achieved and the value function is continuous in ϵ in \mathbb{R}_+ .

Observe that, when ϵ in \mathbb{R}_+ is sufficiently large, the truncated problem delivers a constrained Pareto efficient allocation, that is, $V_\theta^t(\epsilon) = \max_{x \in X(e)} W_\theta(x)$; when ϵ in \mathbb{R}_+ vanishes, the truncated problem delivers the autarchy, as this is the only feasible allocation x in $X(e)$ satisfying additional constraints (\dagger) , that is, $V_\theta^t(0) = W_\theta(e)$. Hence, by the Intermediate Value Theorem, for some value of ϵ in \mathbb{R}_+ ,

$$V_\theta^t(\epsilon) = W_\theta(e) + \xi \left(\max_{x \in X(e)} W_\theta(x) - W_\theta(e) \right).$$

Let x^t be an allocation in $X(e)$ that solves the t -truncated planner problem at the value of ϵ in \mathbb{R}_+ fulfilling the above restriction. \square

Limit. The sequence of allocation $\{x^t\}_{t \in \mathcal{T}}$ in $X(e)$, at no loss of generality, converges to some allocation x in $X(e)$ in the product topology. Also, by continuity of preferences, restriction (w-1) is satisfied by the limit allocation x in $X(e)$. \square

Malinvaud optimality in the limit. It remains to verify that condition (w-2) is valid in the limit. Supposing not, there exists an alternative allocation z in $C(e, x)$ satisfying $W_\theta(z) > W_\theta(x)$. Defining the set \mathcal{F} as in the previous part, for every sufficiently large t in \mathcal{T} , $\mathcal{F} \subset \mathcal{S}^t$. Hence, allocation z in $X(e)$ fulfills the additional restrictions (\dagger) for every sufficiently large t in \mathcal{T} . It follows that, for every sufficiently large t in \mathcal{T} , $W_\theta(x^t) \geq W_\theta(z)$, yielding a contradiction. \square

The sequence of steps proves the proposition. \square

Proof of proposition 2. Using lemma 2, Malinvaud efficiency follows from the simple first-order characterization of equilibrium that is provided by Alvarez and Jermann [4, Propositions 4.5-4.6]. \square

Proof of proposition 3. The proof is rather involved, so that we decompose it in several steps.

Recovering financial plans. To simplify notation, we introduce the positive linear operator $T : L \rightarrow L$ that is defined, at every date-event σ in \mathcal{S} , by

$$T(v)_\sigma = \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau.$$

For an individual i in \mathcal{J} , let g^i in L be given, at every date-event σ in \mathcal{S} , by

$$g_\sigma^i = \left(\frac{\pi_\sigma^i}{p_\sigma^i} \right) (U_\sigma^i(x^i) - U_\sigma^i(e^i)) \leq \frac{1}{\pi_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} \pi_\tau^i \left(\frac{u^i(1/\epsilon) - u^i(\epsilon)}{\partial u^i(1/\epsilon)} \right).$$

Notice that, by uniform impatience and feasibility, g^i is a bounded and positive element of L . Furthermore, observe that, by convexity of preferences, at every date-event σ in \mathcal{S} ,

$$\begin{aligned} \pi_\sigma^i (U_\sigma^i(x^i) - U_\sigma^i(e^i)) &\geq \pi_\sigma^i (u^i(x_\sigma^i) - u^i(e_\sigma^i)) + \sum_{\tau \in \sigma_+} \pi_\tau^i (U_\tau^i(x^i) - U_\tau^i(e^i)) \\ &\geq p_\sigma^i (x_\sigma^i - e_\sigma^i) + \sum_{\tau \in \sigma_+} \pi_\tau^i (U_\tau^i(x^i) - U_\tau^i(e^i)). \end{aligned}$$

This, exploiting first-order conditions at a Malinvaud optimum, yields

$$T(g^i) + (x^i - e^i) \leq g^i.$$

Finally, define $g = \sum_{i \in \mathcal{J}} g^i$ and observe that g is a positive bounded element of L .

Define H as the set of all h in $[0, g]^\mathcal{J}$ satisfying

$$\sum_{i \in \mathcal{J}} h^i = g.$$

The set H is non-empty, convex and compact (in the product topology). Define a correspondence $f : H \rightarrow H$ by means of

$$f(h)_\sigma = \arg \min_{h_\sigma \in H_\sigma} \sum_{i \in \mathcal{J}} \hat{h}_\sigma^i (T(g^i - h^i)_\sigma + (x^i - e^i)_\sigma - (g^i - h^i)_\sigma).$$

Basically, if a financial plan lies in the interior of the budget constraint at some date-event, current debt is increased. By construction, given any h in H , at every date-event σ in \mathcal{S} , there exists an individual i in \mathcal{J} such that

$$T(g^i - h^i)_\sigma + (x^i - e^i)_\sigma \leq (g^i - h^i)_\sigma,$$

as

$$\sum_{i \in \mathcal{J}} (T(g^i - h^i) + (x^i - e^i)) = 0 = \sum_{i \in \mathcal{J}} (g^i - h^i).$$

The correspondence $f : H \rightarrow H$ is clearly closed with non-empty and convex values. Moreover, since H is compact, f is also upper hemi-continuous. Therefore, by Kakutani Fixed Point Theorem, it admits a fixed point h in H . At a fixed point, for every individual i in \mathcal{J} , at any date-event σ in \mathcal{S} ,

$$T(g^i - h^i)_\sigma + (x^i - e^i)_\sigma > (g^i - h^i)_\sigma \text{ implies } h_\sigma^i = 0.$$

Hence,

$$g_\sigma^i \geq T(g^i)_\sigma + (x^i - e^i)_\sigma \geq T(g^i - h^i)_\sigma + (x^i - e^i)_\sigma > g_\sigma^i,$$

which is a contradiction. Thus, at a fixed point, for every individual i in \mathcal{J} ,

$$T(g^i - h^i) + (x^i - e^i) \leq (g^i - h^i).$$

Since, aggregating on \mathcal{J} , $\sum_{i \in \mathcal{J}} h^i = \sum_{i \in \mathcal{J}} g^i$, the latest suffices to prove exact budget-feasibility, that is, for every individual i in \mathcal{J} ,

$$T(g^i - h^i) + (x^i - e^i) = (g^i - h^i).$$

To conclude, for every individual i in \mathcal{J} , the financial plan $v^i = g^i - h^i$ in V^i is bounded, balances budget sequentially and satisfies, at every date-event σ in \mathcal{S} ,

$$U_\sigma^i(x^i) = U_\sigma^i(e^i) \text{ only if } v_\sigma^i = g_\sigma^i - h_\sigma^i \leq g_\sigma^i \leq 0.$$

Furthermore, across individuals, financial plans v in V satisfy market clearing, that is,

$$\sum_{i \in \mathcal{J}} v^i = 0.$$

We treat such financial plans as given in the remaining parts of this proof. \square

Individual optimality. For every individual i in \mathcal{J} , consider the set of all date-events at which this individual is at the autarchic utility, that is,

$$\mathcal{S}^i = \{\sigma \in \mathcal{S} : U_\sigma^i(x^i) = U_\sigma^i(e^i)\}.$$

Also, define the space $F^i(x^i, v^i)$ of all debt limits f^i in F^i satisfying, at every date-event σ in \mathcal{S} , $v_\sigma^i + f_\sigma^i \geq 0$ and, at every date-event σ in \mathcal{S}^i , $v_\sigma^i + f_\sigma^i = 0$. We here show that consumption plan x^i in X^i is optimal, subject to budget and debt constraints, given initial claims, at all debt limits f^i in $F^i(x^i, v^i)$.

Peg any date-event σ in \mathcal{S} . Observe that, as budget is balanced,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i - \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (v_\tau^i + f_\tau^i) - (x_\sigma^i - e_\sigma^i) \leq -v_\sigma^i.$$

Furthermore, considering any alternative budget feasible consumption plan z^i in X^i satisfying debt constraints,

$$-\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i + \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (w_\tau^i + f_\tau^i) + (z_\sigma^i - e_\sigma^i) \leq w_\sigma^i.$$

Using first-order conditions, one obtains

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i - \frac{1}{p_\sigma^i} \sum_{\tau \in \sigma_+} p_\tau^i (v_\tau^i + f_\tau^i) - (x_\sigma^i - e_\sigma^i) \leq -v_\sigma^i$$

and

$$-\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i + \frac{1}{p_\sigma^i} \sum_{\tau \in \sigma_+} p_\tau^i (w_\tau^i + f_\tau^i) + (z_\sigma^i - e_\sigma^i) \leq w_\sigma^i.$$

Therefore, adding up terms, it follows that

$$(\dagger\dagger) \quad \sum_{\tau \in \sigma_+} p_\tau^i (w_\tau^i - v_\tau^i) + p_\sigma^i (z_\sigma^i - x_\sigma^i) \leq p_\sigma^i (w_\sigma^i - v_\sigma^i).$$

For every t in \mathcal{T} , let

$$\mathcal{S}_t = \{\sigma \in \mathcal{S} : t(\sigma) = t\} \text{ and } \mathcal{S}^t = \{\sigma \in \mathcal{S} : t(\sigma) \leq t\}.$$

Consolidating inequalities $(\dagger\dagger)$ up to period t in \mathcal{T} , and using the fact the initial claims are given,

$$\sum_{\sigma \in \mathcal{S}^t} p_\sigma^i (z_\sigma^i - x_\sigma^i) \leq \sum_{\tau \in \mathcal{S}_{t+1}} p_\tau^i (v_\tau^i - w_\tau^i) \leq \sum_{\tau \in \mathcal{S}_{t+1}} p_\tau^i (v_\tau^i + f_\tau^i),$$

where the last inequality follows from debt constraints. By concavity of utility, this suffices to prove optimality, as the right hand-side vanishes in the limit, because p^i in P^i is a summable element of L and $v^i + f^i$ is a bounded element of L . \square

Recovering debt limits. Given debt limits f^i in F^i , at every date-event $\bar{\sigma}$ in \mathcal{S} , let $B_{\bar{\sigma}}^i(w_{\bar{\sigma}}^i; f^i)$ be the set of all plans (\bar{x}^i, \bar{v}^i) in $X^i \times V^i$ satisfying, at every date-event σ in $\mathcal{S}(\bar{\sigma})$, budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma (\bar{x}_\sigma^i - e_\sigma^i) \leq p_\sigma \bar{v}_\sigma^i,$$

and debt constraints,

$$-(\bar{v}_\tau^i + f_\tau^i)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth w_σ^i in \mathbb{R} .

Consider the set

$$D^i = \{(w^i, f^i) : B_{\bar{\sigma}}^i(w_{\bar{\sigma}}^i; f^i) \text{ is non-empty at every } \bar{\sigma} \in \mathcal{S}\} \subset V^i \times F^i.$$

This domain is non-empty, closed and convex. Define a value function $J^i : D^i \rightarrow L$ by means, at every date event $\bar{\sigma}$ in \mathcal{S} , of

$$J_{\bar{\sigma}}^i(w_{\bar{\sigma}}^i; f^i) = \max \{U_{\bar{\sigma}}^i(\bar{x}^i) : (\bar{x}^i, \bar{v}^i) \in B_{\bar{\sigma}}^i(w_{\bar{\sigma}}^i; f^i)\}.$$

It is straightforward to verify that this value function is well-defined, as the maximum x^i is achieved, and fulfils the following properties: (i) it is bounded; (ii) it is concave; (iii) it is weakly increasing in f^i in F^i and strictly increasing in w^i in V^i on its domain D^i ; (iv) for every \bar{f}^i in F^i , it is continuous on the restricted domain

$$\{(w^i, f^i) \in D^i : f^i = \bar{f}^i\}$$

and upper semi-continuous on the restricted domain

$$\{(w^i, f^i) \in D^i : f^i \leq \bar{f}^i\};$$

(v) for every f^i in $F^i(x^i, v^i)$, by construction, (v^i, f^i) is an element of the domain D^i and, by the previous argument for optimality, at every date-event $\bar{\sigma}$ in \mathcal{S} ,

$$(\diamond) \quad J_{\bar{\sigma}}^i(v_{\bar{\sigma}}^i; f^i) = U_{\bar{\sigma}}^i(x^i).$$

We now show some properties of differentiability of the value function. Given any f^i in $F^i(x^i, v^i)$ and any w^i in V^i satisfying $w^i \geq v^i - x^i$, (w^i, f^i) is an element of the domain D^i and, by optimality, at every date-event σ in \mathcal{S} ,

$$\begin{aligned} J_{\sigma}^i(w_{\sigma}^i; f^i) &\geq u^i(x_{\sigma}^i + (w_{\sigma}^i - v_{\sigma}^i)) + \frac{1}{\pi_{\sigma}^i} \sum_{\tau \in \sigma_+} \pi_{\tau}^i J_{\tau}^i(v_{\tau}^i; f^i) \\ &\geq u^i(x_{\sigma}^i + (w_{\sigma}^i - v_{\sigma}^i)) + \frac{1}{\pi_{\sigma}^i} \sum_{\tau \in \sigma_+} \pi_{\tau}^i U_{\tau}^i(x^i). \end{aligned}$$

By a well-known result in convex analysis (see Benveniste and Sheinkman [4]), given any f^i in $F^i(x^i, v^i)$, the value function admits a (partial) derivative at $(v^i; f^i)$ in D^i and, at every date-event σ in \mathcal{S} ,

$$\partial J_{\sigma}^i(v_{\sigma}^i; f^i) = \partial u^i(x_{\sigma}^i)$$

Thus, given any f^i in $F^i(x^i, v^i)$, consider any (w^i, f^i) in D^i satisfying, at some date-event σ in \mathcal{S} , $J_{\sigma}^i(w_{\sigma}^i; f^i) \geq U_{\sigma}^i(e^i)$ and $w_{\sigma}^i \leq v_{\sigma}^i$. Concavity delivers

$$\begin{aligned} U_{\sigma}^i(e^i) - U_{\sigma}^i(x^i) &\leq \\ J_{\sigma}^i(w_{\sigma}^i; f^i) - J_{\sigma}^i(v_{\sigma}^i; f^i) &\leq \partial u^i(x_{\sigma}^i)(w_{\sigma}^i - v_{\sigma}^i) \\ &\leq \xi(w_{\sigma}^i - v_{\sigma}^i), \end{aligned}$$

where

$$\xi = \bigwedge_{\sigma \in \mathcal{S}} \partial u^i(x_{\sigma}^i) > 0.$$

Therefore, rearranging terms,

$$v_{\sigma}^i - \frac{U_{\sigma}^i(x^i) - U_{\sigma}^i(e^i)}{\xi} \leq w_{\sigma}^i.$$

Also, by uniform impatience and boundedness of per-period utility, there exists a sufficiently large $\phi > 0$ satisfying

$$\phi > \bigvee_{\sigma \in \mathcal{S}} \frac{U_{\sigma}^i(x^i) - U_{\sigma}^i(e^i)}{\xi}.$$

It follows that, given any f^i in $F^i(x^i, v^i)$, for every (w^i, f^i) in D^i ,

$$(\ddagger) \quad J_{\sigma}^i(w_{\sigma}^i; f^i) \geq U_{\sigma}^i(e^i) \text{ only if } w_{\sigma}^i \geq v_{\sigma}^i - \phi.$$

We shall exploit this fundamental inequality in order to recover not-too-tight debt limits.

We implicitly define an operator $G^i : F^i(x^i, v^i) \rightarrow F^i(x^i, v^i)$ by setting, at every date-event σ in \mathcal{S} ,

$$J_\sigma^i(-G^i(f^i)_\sigma; f^i) = U_\sigma^i(e^i).$$

To prove that this operator is well defined, pegging any date-event σ in \mathcal{S} , observe that

$$J_\sigma^i(0; f^i) \geq J_\sigma^i(0; 0) \geq U_\sigma^i(e^i)$$

and

$$J_\sigma^i(-g_\sigma^i; f^i) \leq u^i(0) + \frac{1}{\pi_\sigma^i} \sum_{\tau \in \sigma_+} \pi_\tau^i U_\tau^i(x^i) < U_\sigma^i(e^i),$$

where

$$0 \leq g_\sigma^i = \sup \{ -w_\sigma^i \in \mathbb{R} : B_\sigma^i(w_\sigma^i; f^i) \text{ is non-empty} \} \leq e_\sigma^i + \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i,$$

as utility satisfies boundary conditions. Hence, by the Intermediate Value Theorem, $G^i(f^i)$ exists in L . Also, it is positive and bounded, as market clearing and first-order conditions (C-1)-(C-2) imply

$$\begin{aligned} \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau f_\tau^i &\leq \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau \sum_{j \in \mathcal{J}} f_\tau^j \\ &= \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau \sum_{j \in \mathcal{J}} (v_\tau^j + f_\tau^j) \\ &= \sum_{j \in \mathcal{J}} \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau (v_\tau^j + f_\tau^j) \\ &= \sum_{j \in \mathcal{J}} \frac{1}{p_\sigma^j} \sum_{\tau \in \sigma_+} p_\tau^j (v_\tau^j + f_\tau^j) \\ &\leq \sum_{j \in \mathcal{J}} \left(\bigvee_{\tau \in \mathcal{S}} |v_\tau^j + f_\tau^j| \right) \frac{1}{p_\sigma^j} \sum_{\tau \in \sigma_+} p_\tau^j \end{aligned}$$

and uniform impatience yields

$$\frac{1}{p_\sigma^j} \sum_{\tau \in \sigma_+} p_\tau^j \leq \frac{1}{\pi_\sigma^j} \sum_{\tau \in \sigma_+} \pi_\tau^j \left(\frac{\partial u^j(\epsilon)}{\partial u^j(1/\epsilon)} \right) \leq \frac{1}{\eta} \left(\frac{\partial u^j(\epsilon)}{\partial u^j(1/\epsilon)} \right).$$

Furthermore, at every date-event σ in \mathcal{S} ,

$$J_\sigma^i(v_\sigma^i; f^i) = U_\sigma^i(x^i) \geq U_\sigma^i(e^i) = J_\sigma^i(-G^i(f^i)_\sigma; f^i)$$

and, at every date-event σ in \mathcal{S}^i ,

$$J_\sigma^i(v_\sigma^i; f^i) = U_\sigma^i(x^i) = U_\sigma^i(e^i) = J_\sigma^i(-G^i(f^i)_\sigma; f^i).$$

Hence, $G^i(f^i)$ is an element of $F^i(x^i, v^i)$. Finally, observe that the operator $G^i : F^i(x^i, v^i) \rightarrow F^i(x^i, v^i)$ is (weakly) monotone.

Construct debt limits \bar{f}^i in $F^i(x^i, v^i)$ so that $\bar{f}_\sigma^i = -v_\sigma^i$, at every date-event σ in \mathcal{S}^i , and $\bar{f}_\sigma^i \geq -v_\sigma^i + \phi$, at every date-event σ in $(\mathcal{S}/\mathcal{S}^i)$. We claim that $G^i(\bar{f}^i)$ in $F^i(x^i, v^i)$ satisfies $G^i(\bar{f}^i) \leq \bar{f}^i$. Indeed, exploiting restriction (\ddagger) , at every date-event σ in $(\mathcal{S}/\mathcal{S}^i)$,

$$G^i(\bar{f}^i)_\sigma \leq -v_\sigma^i + \phi \leq \bar{f}_\sigma^i;$$

at every date-event σ in \mathcal{S}^i ,

$$G^i(\bar{f}^i)_\sigma = -v_\sigma^i = \bar{f}_\sigma^i.$$

Now, by induction, construct a sequence $((G^i)^n(\bar{f}^i))_{n \in \mathcal{T}}$ in $F^i(x^i, v^i)$. Such a sequence is weakly decreasing and bounded, as

$$\bar{f}^i \geq (G^i)^n(\bar{f}^i) \geq (G^i)^{n+1}(\bar{f}^i) \geq -v^i.$$

Hence, it converges to some f^i in $F^i(x^i, v^i)$ in the product topology. By upper semi-continuity of the value function, at every date-event σ in \mathcal{S} ,

$$J_\sigma^i(-f_\sigma^i; f^i) \geq U_\sigma^i(e^i).$$

Finally, suppose the latest holds with strict inequality at some date-event σ in \mathcal{S} , *i.e.*, there exists $\epsilon > 0$ such that

$$J_\sigma^i(-f_\sigma^i; f^i) > J_\sigma^i(-f_\sigma^i - \epsilon; f^i) > U_\sigma^i(e^i).$$

For every sufficiently large n in \mathcal{T} , $(G^i)^{n+1}(\bar{f}^i)_\sigma \leq f_\sigma^i + \epsilon$ and, therefore,

$$\begin{aligned} U_\sigma^i(e^i) &\geq J_\sigma^i(-(G^i)^{n+1}(\bar{f}^i)_\sigma; (G^i)^n(\bar{f}^i)) \\ &\geq J_\sigma^i(-f_\sigma^i - \epsilon; (G^i)^n(\bar{f}^i)) \\ &\geq J_\sigma^i(-f_\sigma^i - \epsilon; f^i) \\ &> U_\sigma^i(e^i), \end{aligned}$$

a contradiction. Hence, f^i in F^i are not-too-tight debt limits at equilibrium. \square

The proof is now complete. \square

Proof of lemma 4. Let $M(e)$ be the set of Malinvaud efficient allocations. The proof of the Second Welfare Theorem implicitly establishes the existence of a correspondence $G : M(e) \hookrightarrow \Omega$ giving distributions of initial claims which are consistent with a given equilibrium allocation. Such a correspondence is closed with non-empty convex values. In addition, at no loss of generality, for some sufficiently large $\mu > 0$, we can truncate Ω as

$$\Gamma = \left\{ \omega \in \Omega : \sum_{i \in \mathcal{I}} |\omega^i| \leq \mu \right\},$$

which is a compact convex set. Indeed, we identify uniform upper bounds for initial claims in the part on recovering financial plans of the proof of proposition 3. Hence, by the Closed Graph Theorem, the correspondence $G : M(e) \hookrightarrow \Gamma$ is upper hemi-continuous with non-empty convex values.

Let

$$f_\theta(e, y) = \arg \max_{z \in F(e, y)} W_\theta(z).$$

By Berge's Maximum Theorem, the correspondence $f(e, y) : \Theta \hookrightarrow F(e, y)$ is upper hemi-continuous with non-empty values. Suppose that $\{x, z\} \subset f_\theta(e, y)$. For every $1 > \lambda > 0$, allocation $\lambda x + (1 - \lambda)z$ lies in $F(e, y)$. By convexity of preferences, it follows that

$$\sum_{i \in \mathcal{I}} \theta^i [U^i(\lambda x^i + (1 - \lambda)z^i) - \lambda U^i(x^i) - (1 - \lambda)U^i(z^i)] = 0.$$

Thus, for every individual i in $\{i \in \mathcal{I} : \theta^i > 0\}$, $x^i = z^i$. Suppose that, for some individual i in $\{i \in \mathcal{I} : \theta^i = 0\}$, $x^i \neq z^i$. This implies that

$$U^i(\lambda x^i + (1 - \lambda)z^i) > \lambda U^i(x^i) + (1 - \lambda)U^i(z^i) \geq U^i(e^i).$$

As a slight reduction of initial consumption of this individual does not affect participation, and such resources can be distributed to individuals in $\{i \in \mathcal{I} : \theta^i > 0\}$, this produces a contradiction. It follows that $f(e, y) : \Theta \hookrightarrow F(e, y)$ is an upper hemi-continuous correspondence with single values and, in fact, a continuous function. Hence, $M(e, y)$ is connected, being the image of a connected set by a continuous function. Thus, $G(M(e, y))$ is connected, being the image of a connected set by an upper hemi-continuous correspondence with non-empty convex values in a topological vector space.

Construct now a correspondence $F : \Theta \times \Gamma \hookrightarrow \Theta \times \Gamma$ by setting, at every $(\bar{\theta}, \bar{\omega})$ in $\Theta \times \Gamma$,

$$F_\Theta(\bar{\theta}, \bar{\omega}) = \arg \min_{\theta \in \Theta} \theta \cdot \bar{\omega} \text{ and } F_\Gamma(\bar{\theta}, \bar{\omega}) = G(f_{\bar{\theta}}(e, y)).$$

This correspondence is upper hemi-continuous with non-empty convex values. By Kakutani's Fixed Point Theorem, it admits a fixed point $(\bar{\theta}, \bar{\omega})$ in $\Theta \times \Gamma$.

Suppose that, at the fixed point, for some individual i in \mathcal{I} , $\bar{\omega}^i > 0$. This implies that $\bar{\theta}^i = 0$ and, by optimality, $U^i(f_{\bar{\theta}}(e, y)^i) = U^i(e)$. This is a contradiction, as a strictly positive initial claim permits to achieve a utility strictly higher than the autarchic utility, subject to budget feasibility, at positive debt limits. Hence, $\bar{\omega} = 0$, which proves the claim. \square

Proof of proposition 4. Given strictly positive welfare weights θ in Θ , we exploit proposition 1 to peg any sequence of allocations $(y^n)_{n \in \mathbb{N}}$ in $X(e)$ satisfying, at every

n in \mathbb{N} ,

$$W_\theta(y^n) = W_\theta(e) + \left(\frac{1}{n}\right) \left(\max_{z \in X(e)} W_\theta(z) - W_\theta(e) \right)$$

and

$$W_\theta(y^n) = \max_{z \in F(e, y^n)} W_\theta(z).$$

Notice that this sequence consists of distinct non-autarchic allocations. Applying lemma 4, there exists a sequence of non-autarchic allocations $(x^n)_{n \in \mathbb{N}}$ in $E_0(e)$ such that, for every n in \mathbb{N} , x^n lies in $M(e, y^n)$. Furthermore, by construction, for every n in \mathbb{N} ,

$$W_\theta(x^n) \leq W_\theta(e) + \left(\frac{1}{n}\right) \left(\max_{z \in X(e)} W_\theta(z) - W_\theta(e) \right).$$

It follows that the sequence $(x^n)_{n \in \mathbb{N}}$ in $E_0(e)$ consists of infinitely many distinct allocations, for otherwise some subsequence would converge to an allocation x in $X(e)$ with $W_\theta(x) > W_\theta(e)$, violating the above inequality. \square

APPENDIX B. SEVERE PUNISHMENTS

By reducing the risk of default, severe punishments enlarge insurance opportunities and increase social welfare at equilibrium. More relevantly for our analysis, they guarantee constrained efficiency of competitive equilibrium and, thus, eliminate indeterminacy.

Assume that the punishment for default consists of the exclusion from financial markets and, in addition, a partial confiscation of private endowment. This hypothesis can be accommodated in our analysis by modifying participation for constrained efficiency and by adapting restrictions for not-too-tight debt limits at equilibrium. In particular, let $1 > \lambda \geq 0$ represent the portion of unexpropriable private endowment after debt repudiation. For every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} , participation now imposes

$$U_\sigma^i(x^i) \geq U_\sigma^i(\lambda e^i).$$

Futhermore, at a competitive equilibrium, debt limits satisfy

$$J_\sigma^i(-f_\sigma^i; f^i) = U_\sigma^i(\lambda e^i).$$

We argue that competitive equilibrium exhibits high implied interest rates (*i.e.*, a finite present value of intertemporal endowment), so guaranteeing that a constrained efficient allocation of risk is achieved (by a straightforward adaptation of Alvarez and Jermann's [4, Corollary 4.7] First Welfare Theorem).

Suppose that, for some individual i in \mathcal{J} , at date-event σ in \mathcal{S} ,

$$p_\sigma f_\sigma^i < (1 - \lambda) p_\sigma e_\sigma^i + \sum_{\tau \in \sigma_+} p_\tau f_\tau^i.$$

This means that, beginning with the maximum amount of debt and issuing future contingent debt up to the limit, a consumption level strictly above the unexpropriable endowment is affordable. Therefore, by strict monotonicity of preferences,

$$\begin{aligned}
J_\sigma^i(-f_\sigma^i; f^i) &> u^i(\lambda e_\sigma^i) + \frac{1}{\pi_\sigma^i} \sum_{\tau \in \sigma_+} \pi_\tau^i J_\tau^i(-f_\tau^i; f^i) \\
&= u^i(\lambda e_\sigma^i) + \frac{1}{\pi_\sigma^i} \sum_{\tau \in \sigma_+} \pi_\tau^i U_\tau^i(\lambda e^i) \\
&= U_\sigma^i(\lambda e^i),
\end{aligned}$$

which is a contradiction. Hence, for every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} ,

$$p_\sigma f_\sigma^i \geq (1 - \lambda) p_\sigma e_\sigma^i + \sum_{\tau \in \sigma_+} p_\tau f_\tau^i.$$

Consolidating over date-events, and exploiting *positivity* of debt limits,

$$f_\sigma^i \geq (1 - \lambda) \frac{1}{p_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau e_\tau^i.$$

This shows that, at equilibrium, prices involve high implied interest rates.

Incidentally, notice that, when the entire endowment can be confiscated after default ($\lambda = 0$), for every individual i in \mathcal{J} , at every date-event σ in \mathcal{S} , debt limits satisfy

$$p_\sigma f_\sigma^i = p_\sigma e_\sigma^i + \sum_{\tau \in \sigma_+} p_\tau f_\tau^i.$$

Thus, by consolidation,

$$f_\sigma^i = \frac{1}{p_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau e_\tau^i + b_\sigma^i,$$

where b^i is a positive element of L satisfying the martingale property

$$p_\sigma b_\sigma^i = \sum_{\tau \in \sigma_+} p_\tau b_\tau^i.$$

This bubble component vanishes at equilibrium by canonical arguments. Hence, *natural debt limits* (that is, debt is bounded by the present value of private endowment) correspond to not-too-tight debt limits with full confiscation of private endowment after default.

APPENDIX C. EXAMPLE

In this example, we examine limited commitment equilibria in the classical cyclic economy of Bewley, also studied by Azariadis, Antinolfi and Bullard [6] and Azariadis [7]. Individuals trade securities to smooth idiosyncratic endowment fluctuations. In a full commitment equilibrium, each individual will consume a constant

share of aggregate endowment. Conversely, in a limited commitment equilibrium, individual consumption might still fluctuate, as complete risk-sharing would violate participation and, hence, induce default.

The economy has no uncertainty, $\mathcal{T} = \mathcal{S}$, and only two individuals called *even* and *odd*, $\mathcal{J} = \{e, o\}$. Preferences of individuals are represented by

$$U(x^i) = \sum_{t \in \mathcal{T}} \beta^t u(x_t^i),$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the common per-period utility and $0 < \beta < 1$ is the common discount factor. Endowments (e^e, e^o) are given by the two sequences

$$\begin{aligned} e^e &= (e_h, e_l, e_h, e_l, e_h, e_l, \dots), \\ e^o &= (e_l, e_h, e_l, e_h, e_l, e_h, \dots), \end{aligned}$$

where $e_h > e_l > 0$. Hence, aggregate endowment is constant and individual endowments are perfectly negatively correlated. Finally, in order to ensure that trade occurs at a constrained optimum and that an unconstrained optimum is not achieved, we postulate that

$$\beta u'(e_l) > u'(e_h)$$

and

$$u\left(\frac{e_h + e_l}{2}\right) + \beta u\left(\frac{e_h + e_l}{2}\right) > u(e_h) + \beta u(e_l).$$

We consider Malinvaud efficient allocations where the high-endowment trader is constrained in every period (which is a necessary property apart from an initial transitory phase). Denoting $(\xi_t)_{t \in \mathcal{T}}$ the sequence of positive transfers from the high-endowment individual to the low-endowment individual, this is restricted by

$$u(e_h - \xi_t) + \beta u(e_l + \xi_{t+1}) = u(e_h) + \beta u(e_l).$$

The obtained implicit difference equation admits a steady state $\xi^* > 0$ and a continuum of other solutions, monotonically decreasing to autarchy, for every initial condition ξ_0 in $(0, \xi^*)$. For each of such allocations, identified with an initial value ξ_0 in $[0, \xi^*]$, the sequence of implicit prices is given by

$$p_{t+1} = \left[\frac{\beta u'(e_l + \xi_{t+1})}{u'(e_h - \xi_t)} \right] p_t.$$

Indeed, market interest rate coincides with the subject interest rate of the unconstrained individual.

To decentralize such Malinvaud efficient allocations as competitive equilibria, let $(v_t)_{t \in \mathcal{T}}$ denote the sequence of positive claims held by an unconstrained individual. By market clearing, the constrained individual holds a positive debt. For a given

Malinvaud efficient allocation ξ_0 in $[0, \xi^*]$, using budget constraint, the equilibrium sequence of claims is restricted by

$$p_{t+1}v_{t+1} = p_t(\xi_t - v_t).$$

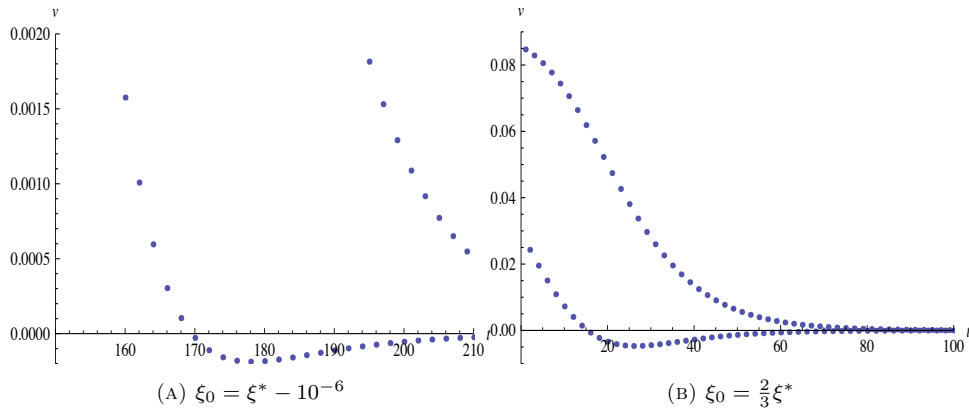
Ignoring positivity, such a difference equation generates a unique solution for any arbitrary initial value v_0 . However, any non-positive solution implies non-positive debt limits, that is, a constrained individual is forced to accumulate assets. Hence, it is not consistent with our restrictive notion of competitive equilibrium.

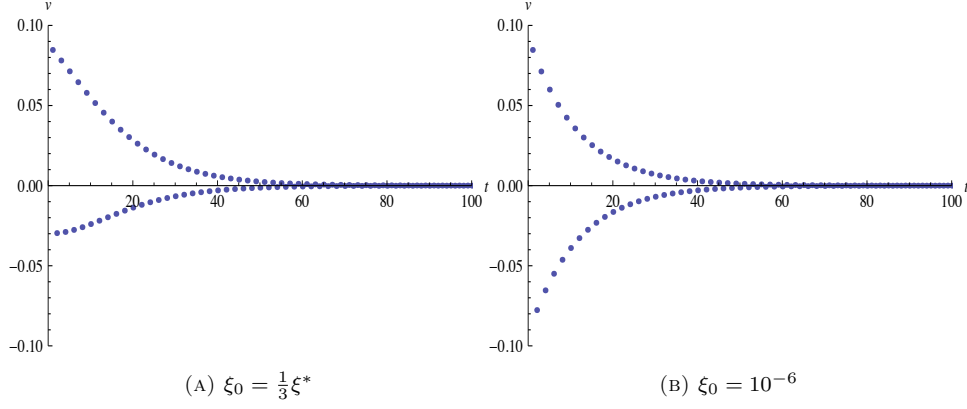
We now perform the following exercise. We peg the initial claim corresponding to the steady state equilibrium with trade,

$$v^* = \left(1 + \frac{\beta u'(e_l + \xi^*)}{u'(e_h - \xi^*)}\right)^{-1} \xi^*.$$

For every allocation ξ_0 in $(0, \xi^*)$, we determine the sequence of claims $(v_t)_{t \in \mathcal{T}}$ consistent with the initial value $v_0 = v^*$. We then verify whether such claims remain positive and, thus, whether they sustain a competitive equilibrium under positive debt limits. Positivity is hard to ascertain by means of analytical methods and, therefore, we rely on numerical computations, although these cannot be decisive on this matter.

Endowments are $(e_l, e_h) = (.7, 1.3)$. The discount factor is $\beta = .8$, whereas the utility function is given by $u(c) = \sqrt{c}$. We find that, after an initial phase, claims (and, hence, debt limits) oscillates between positive and negative values, over some range, before converging to zero. These negative values of debt limits are smaller, and occur in a farer horizon, the closer is the initial condition ξ_0 to efficiency ξ^* . Our grid of initial values ξ_0 is contained in $(0, \xi^*)$ with increments of 10^{-6} . The four panels below report the sequences of claims for different initial conditions ξ_0 in $(0, \xi^*)$.





These computations cast some doubts on the claim of indeterminacy submitted by Azariadis, Antinolfi and Bullard [6] (see, also, Azariadis [7]). It seems that the constrained efficient steady state is determinate and, indeed, the unique equilibrium of an economy with that distribution of initial claims, when debt limits are restricted to be positive. As previously mentioned, all equilibria are indeterminate when debt limits are not constrained by positivity.